

Some remarks on the Pigola-Rigoli-Setti version of the Omori-Yau maximum principle

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Abstract. We prove that the hypotheses in the version of the Omori-Yau maximum principle that was given by Pigola-Rigoli-Setti are logically equivalent to the assumption that the manifold carries a C^2 proper function whose gradient and Hessian (Laplacian) are bounded. In particular, this result extends the scope of the original Omori-Yau principle, formulated in terms of lower bounds for curvature.

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1 Introduction

The celebrated Omori-Yau maximum principle ([3], [5], [8]) states that if (M, g) is a complete Riemannian manifold with sectional curvature (resp. Ricci curvature) bounded below then, for every $f \in C^2(M)$ that is bounded above, there exists a sequence (q_k) in M such that

$$f(q_k) > \sup_M f - \frac{1}{k}, \quad |\nabla f|(q_k) < \frac{1}{k}, \quad \text{Hess} f(q_k) < \frac{1}{k} g(q_k) \quad \left(\text{resp. } \Delta f(q_k) < \frac{1}{k} \right), \quad (1.1)$$

for all $k \in \mathbb{N}$, where the third inequality above is in the sense of quadratic forms.

Pigola-Rigoli-Setti ([6], Theorem 1.9) obtained a version of the Omori-Yau maximum principle where the hypothesis that the curvature is bounded below is replaced by the assumption that the manifold admits a smooth function with special properties. More precisely, they proved the following result:

Theorem 1.1. *Let (M, g) be a Riemannian manifold. Assume that there exist a C^2 function $\gamma : M \rightarrow [0, +\infty)$, a compact set $K \subset M$ and constants $A, B > 0$ such that*

(i) γ is proper, i.e., $\gamma(x) \rightarrow +\infty$ as $x \rightarrow \infty$,

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$$(ii) |\nabla \gamma| \leq A\sqrt{\gamma} \quad \text{on } M \setminus K,$$

$$(iii) \text{Hess } \gamma \leq B\sqrt{\gamma G(\sqrt{\gamma})} g \quad \left(\text{resp. } \Delta \gamma \leq B\sqrt{\gamma G(\sqrt{\gamma})} \right) \quad \text{on } M \setminus K,$$

where $G : [0, +\infty) \rightarrow [0, +\infty)$ is a smooth function satisfying

$$G(0) > 0, \quad G'(t) \geq 0 \text{ for all } t \geq 0, \quad (1.2)$$

$$\int_0^{+\infty} \frac{dt}{\sqrt{G(t)}} = +\infty \quad \text{and} \quad \limsup_{t \rightarrow +\infty} \frac{tG(\sqrt{t})}{G(t)} < +\infty. \quad (1.3)$$

Then, for every $f \in C^2(M)$ that is bounded above, there exists a sequence (q_k) in M satisfying (1.1).

The function theoretic approach to the Omori-Yau maximum principle provided by Theorem 1.1 has been applied by several authors to obtain results in different contexts ([1], [2], [6]). In Section 2 of this work, we show that Theorem 1.1 is logically equivalent to a more conceptual statement:

Theorem 1.2. *If a Riemannian manifold (M, g) admits a C^2 function $\phi : M \rightarrow \mathbb{R}$ satisfying, for some constants $C, D > 0$,*

$$(i) \phi \text{ is proper,}$$

$$(ii) |\nabla \phi| \leq C,$$

$$(iii) \text{Hess } \phi \leq Dg \quad \left(\text{resp. } \Delta \phi \leq D \right),$$

then, for every $f \in C^2(M)$ that is bounded above, there exists a sequence (q_k) in M satisfying (1.1).

Under the assumption that a Riemannian manifold M is complete and has sectional curvature (resp. Ricci curvature) bounded below, Schoen-Yau ([7], Theorem 4.2) proved that, for every $p \in M$, there exists a smooth function $\phi : M \rightarrow \mathbb{R}$ satisfying, for some constants $C, D > 0$,

$$\phi(x) \geq d(x, p), \quad |\nabla \phi|(x) \leq C, \quad \text{Hess } \phi(x)(v, v) \leq D|v|^2 \quad \left(\text{resp. } \Delta \phi(x) \leq D \right),$$

for all $x \in M$ and $v \in T_x M$. In particular, ϕ satisfies (i), (ii) and (iii) in the statement of Theorem 1.2. This shows that Theorem 1.2 (and so Theorem 1.1) is a generalization of the original Omori-Yau maximum principle.

2 The arguments.

The proof below was inspired by the proof of a conceptual refinement of the Omori-Yau maximum principle stated in [4].

Proof of the Theorem 1.2. We will give the proof of the half of the theorem that refers to the Hessian. The proof of the other half is entirely analogous and will be left to the reader.

Multiplying ϕ by a positive constant if necessary, we can assume that $C = D = 1$. Let $f : M \rightarrow \mathbb{R}$ be a C^2 function satisfying $\sup_M f < +\infty$ and (p_k) a sequence in M such that

$$f(p_k) > \sup_M f - \frac{1}{2k}, \quad k \in \mathbb{N}. \quad (2.1)$$

For each $k \in \mathbb{N}$, define a function $f_k : M \rightarrow \mathbb{R}$ by

$$f_k(x) = f(x) - \varepsilon_k[\phi(x) - \phi(p_k)], \quad (2.2)$$

where $\varepsilon_k = \min\{\eta_k, 1/2k\}$ and

$$\eta_k = \begin{cases} 1/2k[\phi(p_k) - \inf_M \phi], & \text{if } \phi(p_k) > \inf_M \phi, \\ 1/2k, & \text{if } \phi(p_k) = \inf_M \phi. \end{cases}$$

Since $\varepsilon_k > 0$ and f is bounded above, from (i) one obtains that $f_k(x) \rightarrow -\infty$ as $x \rightarrow \infty$, and so f_k attains a global maximum at some point $q_k \in M$. Hence, by (2.2) and the ordinary maximum principle,

$$0 = \nabla f_k(q_k) = \nabla f(q_k) - \varepsilon_k \nabla \phi(q_k) \quad (2.3)$$

and

$$0 \geq \text{Hess}(f_k)(q_k)(v, v) = \text{Hess}f(q_k)(v, v) - \varepsilon_k \text{Hess}\phi(q_k)(v, v), \quad \forall v \in T_{q_k}M. \quad (2.4)$$

From (ii), (2.3) and definition of ε_k , one obtains

$$|\nabla f(q_k)| = \varepsilon_k |\nabla \phi(q_k)| \leq \varepsilon_k \leq \frac{1}{2k} < \frac{1}{k}.$$

By (iii) and (2.4), we have, for all $v \in T_{q_k}M$ with $v \neq 0$,

$$\text{Hess}f(q_k)(v, v) \leq \varepsilon_k \text{Hess}\phi(q_k)(v, v) \leq \varepsilon_k |v|^2 \leq \frac{1}{2k} |v|^2 < \frac{1}{k} |v|^2.$$

Since $f_k(p_k) = f(p_k)$, we also have

$$\begin{aligned} f(p_k) &= f_k(p_k) \leq f_k(q_k) = f(q_k) - \varepsilon_k[\phi(q_k) - \phi(p_k)] \\ &= f(q_k) - \varepsilon_k[\phi(q_k) - \inf_M \phi] - \varepsilon_k[\inf_M \phi - \phi(p_k)] \\ &\leq f(q_k) - \varepsilon_k[\inf_M \phi - \phi(p_k)] \leq f(q_k) + \frac{1}{2k}. \end{aligned}$$

Therefore, by (2.1),

$$f(q_k) \geq f(p_k) - \frac{1}{2k} > \sup_M f - \frac{1}{2k} - \frac{1}{2k} = \sup_M f - \frac{1}{k},$$

which completes the proof of the theorem. \square

The fact that Theorem 1.1 is equivalent to Theorem 1.2 is an immediate consequence of the following proposition.

Proposition 2.1. *A Riemannian manifold (M, g) admits a function $\gamma : M \rightarrow \mathbb{R}$ as in the statement of Theorem 1.1 if and only if it admits a function $\phi : M \rightarrow \mathbb{R}$ as in the statement of Theorem 1.2.*

Proof. Let $\gamma : M \rightarrow \mathbb{R}$ be a function as in the statement of Theorem 1.1. Define a (smooth) function $u : (0, +\infty) \rightarrow (0, +\infty)$ by

$$u(t) = \sqrt{tG(\sqrt{t})}.$$

From (1.2), one obtains

$$u'(t) = \frac{G(\sqrt{t}) + \frac{1}{2}\sqrt{t}G'(\sqrt{t})}{2\sqrt{tG(\sqrt{t})}} > 0, \quad t > 0.$$

Given $C > \limsup_{t \rightarrow +\infty} tG(\sqrt{t})/G(t)$, there exists $t_o > 1$ such that

$$\frac{tG(\sqrt{t})}{G(t)} < C, \quad t \geq t_o.$$

From the above inequality and the fact that G is non-decreasing, we obtain

$$0 < tG(0) \leq tG(\sqrt{t}) < CG(t), \quad t \geq t_o,$$

and so

$$\frac{1}{\sqrt{tG(\sqrt{t})}} > \frac{1}{\sqrt{C}\sqrt{G(t)}}, \quad t \geq t_o.$$

Hence

$$\begin{aligned} \int_1^{+\infty} \frac{1}{u(s)} ds &= \int_1^{+\infty} \frac{1}{\sqrt{sG(\sqrt{s})}} ds \geq \int_{t_o}^{+\infty} \frac{1}{\sqrt{sG(\sqrt{s})}} ds \\ &\geq \frac{1}{\sqrt{C}} \int_{t_o}^{+\infty} \frac{1}{\sqrt{G(s)}} ds = +\infty, \end{aligned} \tag{2.5}$$

where in the last equality we used (1.3).

Since $u(t)$ and \sqrt{t} are non-decreasing, we can assume, adding a positive constant if necessary, that $\gamma > 0$ and that (ii) and (iii) in the statement of Theorem 1.1 holds on all of M . Therefore, for all $x \in M$ and all $v \in T_x M$ we have

$$\begin{aligned} |\nabla \gamma(x)| &\leq A\sqrt{\gamma(x)} = \frac{A}{\sqrt{G(0)}} \sqrt{\gamma(x)G(0)} \leq \frac{A}{\sqrt{G(0)}} \sqrt{\gamma(x)G(\sqrt{\gamma(x)})} \\ &\leq \frac{A}{\sqrt{G(0)}} u(\gamma(x)) \end{aligned} \tag{2.6}$$

and

$$\text{Hess } \gamma(x)(v, v) \leq B \sqrt{\gamma(x)G(\sqrt{\gamma(x)})} |v|^2 = Bu(\gamma(x))|v|^2. \quad (2.7)$$

Let $h : (0, +\infty) \rightarrow \mathbb{R}$ be defined by

$$h(t) = \int_1^t \frac{1}{u(s)} ds$$

Since $u > 0$ and $u' > 0$, we have, for all $t > 0$,

$$h'(t) = \frac{1}{u(t)} > 0, \quad h''(t) = -\frac{u'(t)}{u^2(t)} < 0. \quad (2.8)$$

Let $\phi = h \circ \gamma$, so that

$$\phi(x) = h(\gamma(x)) = \int_1^{\gamma(x)} \frac{1}{u(s)} ds, \quad x \in M. \quad (2.9)$$

From (2.6) and (2.8), we have

$$|\nabla \phi(x)| = |h'(\gamma(x))\nabla \gamma(x)| = \frac{1}{u(\gamma(x))} |\nabla \gamma(x)| \leq \frac{A}{\sqrt{G(0)}}, \quad \forall x \in M. \quad (2.10)$$

Using (2.7) and (2.8), we obtain, for all $x \in M$ and all $v \in T_x M$,

$$\begin{aligned} \text{Hess } \phi(x)(v, v) &= h''(\gamma(x))\langle \nabla \gamma(x), v \rangle^2 + h'(\gamma(x))\text{Hess } \gamma(x)(v, v) \\ &\leq \frac{1}{u(\gamma(x))}\text{Hess } \gamma(x)(v, v) \leq B|v|^2. \end{aligned} \quad (2.11)$$

Moreover, from (2.5), (2.9) and the properness of γ , we obtain that ϕ is proper. This concludes the proof of the “only if” part of the proposition. The “if” part is easy and will be left to the reader. \square

Remark 2.2. *The above proof shows that Theorem 1.1 (and so Theorem 1.2) is equivalent to saying that the Omori-Yau maximum principle holds on every Riemannian manifold (M, g) that carries a positive proper C^2 function γ satisfying, outside a compact set,*

$$|\nabla \gamma| \leq u \circ \gamma \quad \text{and} \quad \text{Hess } \gamma \leq (u \circ \gamma)g \quad (\text{resp. } \Delta \gamma \leq u \circ \gamma),$$

where $u \in C^1(0, +\infty)$ is a positive function with $u' \geq 0$ and $\int_1^{+\infty} u(s)^{-1} ds = +\infty$.

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